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EFFECT OF SOME MATRIX PROPERTIES OF ELECTRIC NETWORKS UPON COMPUTER SOLUTIONS

by

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I. INTRODUCTION

A. Purpose

The purpose of this investigation is to derive some of the matrix properties of electric networks, especially those networks intended for the transmission of power, and to apply the methods of matrix calculus to the solution of such networks. Although the methods described are applicable to any linear network, it is anticipated that they will be most useful in the study of power systems.

Up to the present time the power system problem has been attacked using primarily the known tools of circuit analysis with the expression of the problem in mathematical terms somewhat suppressed. This has appeared to be the most convenient and direct means available to solve such problems. However, as will be shown later, it is helpful to view the problem as simply a system of equations for which a solution is desired. Here the methods of matrix calculus are particularly powerful because of the simplicity with which long and complicated mathematical statements can be compressed using these techniques.

It is also enlightening to set down the boundary conditions of the physical problem as mathematical statements. From this point it is possible to clearly determine the steps required to solve many problems involving electric networks which are not usually attempted because of their complexity.

B. General Description of the Power System Problem

Because the problem under consideration has the power system as its origin, a few remarks concerning the modern power system are required. The paths available for the flow of electric energy from generator to consumer are becoming increasingly numerous and complex in the modern power system. This is due primarily to the economy inherent in very large generating facilities located in low fuel cost areas, often at large distances from load centers. This requires large networks of transmission lines to carry power to consumer areas and, to guard against loss of generating facilities, requires strong transmission ties between generating stations. This pattern has intensified over the years with the growth of both loads and generating stations. The transmission networks have become quite large in many cases and the solution of these networks by mathematical means has become increasingly difficult. Before digital computers were available these problems were usually solved on large scale analog computers or network analyzers which were designed specifically for this purpose. Now, with digital computers in wide use it is only natural that solutions be attempted which will utilize the speed and accuracy of these devices.

II. REVIEW OF LITERATURE

Beginning in 1946, articles began appearing in the literature occasionally regarding the use of the digital computer or the "business machine" of that day in the solution of electric networks (1,2,3). Up to that time, problems involving power systems were usually solved on network analyzers and little thought was concentrated on strictly mathematical solutions of these systems. However, the early articles by Dunstan (2,3,4) aroused much interest and the years that followed produced several new methods for handling these problems. This evolution of computer usage has been accompanied by a closer examination of the mathematics of the problem and this in turn has brought forth several significant papers.

In order to clarify the historical development of this area a few comments are in order regarding general methods used by other investigators. As the methods used to date fall into definite patterns as to type of solution, they will be discussed in this grouping rather than by authors.

A. Early Attempts at Computer Usage

In 1946, a paper appeared (1) which outlined a solution of a power system problem for an accounting or business machine. In this paper the authors did not attempt a problem with non-linear boundary conditions. Instead, they assumed that both load and generator currents were known in magnitude and phase angle. This simplification, although quite unrealistic, reduced the problem to a set of linear equations which could be easily solved by conventional techniques using loop currents. Perhaps the main contribution of this early effort was to arouse the interest of engineers in the use of computers.

B. Loop and Track Methods

Solutions to the power system problem which closely resemble the well known loop current technique were first proposed by Dunstan (2,3,4). Although never widely accepted, these papers contain some clever extensions of loop currents to loop "load loss flows" or power flows. One method (2) converges by a technique referred to as successive approximations in which loop flow estimates are repeatedly improved until boundary conditions are satisfied. Another suggested method (3) requires a matrix inversion in each iteration to solve a loop current analogy, but appears to have a more serious limitation in the use of unrealistic boundary conditions. Despite these difficulties, these methods converge in a very few iterations.

A later paper by Henderson (5) described another loop or track method based upon the earlier work of Dunstan which reduced the number of matrix inversions to one instead of the earlier one per iteration. In this method power and reactive power are used instead of current to satisfy Kirchhoff's voltage law. Briefly, each iteration is carried out as follows: First a flow of real and reactive power is estimated in each line. Then losses are computed and subtracted from the losses of the previous iteration to determine an incremental loss which in turn is applied to each mesh to determine an incremental flow. This incremental flow is superimposed upon the original flow to determine a new flow. Now the voltage and phase shift across each line is determined and loop errors are found which again requires a balancing flow to satisfy Kirchhoff's law.

This balancing flow is superimposed upon the one previously computed and the process is repeated. This ingenious scheme converges to "network analyzer accuracy" in not more than three iterations according to the author. However, in all fairness to other authors it should be made clear that most methods devised are expected to converge to an accuracy several orders of magnitude greater than that obtainable on a network analyzer.

C. The Ward and Hale Method (Nodal Method)

In 1956 a paper was published by Ward and Hale (6) which has since become almost a standard for comparison because of its wide acceptance. This paper, unlike those preceding it, was based upon nodal techniques and was the first paper to describe a power system with its nonlinear boundary conditions adequately. Since it also avoided the matrix inversion required of the loop method it was indeed a major improvement. The method has one serious drawback however since it converges very slowly and therefore requires a large number of iterations unless the initial voltage estimates are very near the solution.

Because of the importance of this paper and its wide use as a basis for comparison of new methods a few detailed comments are in order. This method is especially attractive for computer solution because it involves no matrix inversion and requires no track diagram. Iteration can be continued until all boundary conditions are satisfied to within some arbitrary precision index and need not be cyclic as required by some other methods. It's primary disadvantage is the relatively large number of iterations required.

Briefly the method is carried out as follows. First all bus voltages are estimated and expressed in rectangular form using convenient values such as 1 + j0 for all busses except one. One bus, usually that of the swing generator, can be completely specified and held at its known value. Second, using the equation $\dot{I} = \sum_{m=1}^{n} \dot{Y}_{km}\dot{E}_m$, the total current entering the first bus is calculated and from this the total power entering the bus is found from $P_k = \mathrm{Re}(\dot{E}_k\dot{I}_k^*)$. This power is compared with the specified load or generator flow for that bus. Third, assuming all other voltages remain constant, $\dot{E}_k$ is corrected in a direction which will minimize the error in load P_k and Q_k (or P_k and E_k for a generator). Fourth, the current equation is again applied to find $\dot{I}_{k+1}$ using the corrected value of $\dot{E}_k$ in this calculation. P_{k+1} is computed and compared with a specified value and a new $\dot{E}_{k+1}$ is found. This process is repeated until the voltage corrections are smaller than a specified precision index.

The method is convenient in its use of normally known system quantities such as per unit watts and vars at load busses and per unit watts and voltage magnitudes at generator busses. These are the same boundary conditions normally applied to network analyzer solutions and are usually known by the power system engineer. The method has been found to diverge on some systems with negative reactances such as found many times in three-winding transformer equivalents.

D. Methods of Speeding Convergence

Following the Ward and Hale paper in 1956, several investigators (8, 9,14,16) published methods for speeding convergence of the original nodal technique. One paper (8) presented a wide variety of schemes for accomplishing speed-up but these were mostly trial and error methods without known mathematical analysis and many of them failed. In some cases the results were rather dramatic in spite of the lack of proof and, for the particular system studied, caused reductions in the number of iterations of about six to one. The methods used were mostly variations of the idea that speedup could be accomplished by multiplying the voltage corrections by some factor, say 1.6 or 1.7, and the multiplier was steadily increased until the method failed. Other methods tried by these authors included simultaneous correction of all bus voltages, linear convergence and relaxation methods. The success or failure of all these methods seemed to be somewhat dependent upon the physical system.

Jordan (9) was able to reduce the number of iterations appreciably by applying a modified Seidel relaxation technique. Using the basic Ward and Hale solution, he computed a residual current for each bus and, by dividing the negative of this residual by the diagonal term of the $\dot{Y}$ matrix was able to obtain a correction component for the bus voltage. The method used was not strictly a relaxation since he took the busses in a fixed order rather than selecting the largest residual for each iteration. One unique feature of the method was the iteration of the swing generator voltage which, after convergence, was then corrected to the final desired value by additional iterations. The method, as reported, had been tried only for power and reactive power boundary conditions which would limit its application. One should note also that, since the $\dot{Y}$ matrix is not positive definite the method will fail when a diagonal term is negative as this would force the solution toward a maximum residual (18).

Van Ness, in a paper (14) published in 1959 suggested several improvements on the Ward and Hale technique. Among these was a solution of the problem in polar form for the unknown voltages. This eliminates solving for the voltage magnitude at generator busses where voltage magnitude is usually given as one of the boundary conditions. In the same paper the author outlines other speed-up techniques, some of which are worthy of mention. One method tried was that of holding all voltage corrections until they had been computed for all busses and then applying them simultaneously. This resulted in no improvement. A second method which was more successful was to make the corrections of the second iteration equal to some given fraction, say 0.5, of that of the first iteration, and so forth. This resulted in a definite speed-up in the convergence and is analogous to the arbitrary speed-up method presented earlier by Brown and Tinney (8).

In 1960, a second paper by Van Ness (16) analyzed in terms of eigenvalues the acceleration techniques discussed in his earlier paper. In this analysis he examined a matrix of coefficients derived by equating the unknown voltage and angle corrections, in polar form, to the errors computed in power and reactive power. He then proceeded to show that the largest eigenvalue of this matrix determined the rate of convergence.

E. Acceptance of Computer Solutions

It is worth mentioning, in connection with the problem of digital computer solution of power flow problems, a few of the many publications relating to results of studies performed on various machines using known techniques of the time. A few typical of these reports are listed (10,11, 12,13) for their value in illustrating the comparison of previously published methods and their handling on computers of widely different size. Many more reports similar to these have appeared in the journals and publications of the engineering field and serve to illustrate the wide interest in the problem at hand.

F. Recent Developments

In a 1959 paper (15), Hale and Goodrich presented a different approach to the problem which they refer to as the transfer ratio method. In effect, this method iterates on voltage at busses where the voltage magnitude is specified and iterates on currents at the remaining busses. This results in a dramatic reduction in the number of iterations (from 54 to 10 in one case). The convergence is oscillatory for this method as compared to the Ward and Hale method which is usually monotonic after a few iterations. Furthermore, the transfer ratio method approaches the solution very fast in the first few iterations after which the oscillations become generally quite small. Little analysis has been done as yet on the exact nature of this method but it may prove to be a very important technique, especially when combined with acceleration techniques.

In February 1961 a paper (17) was presented by Van Ness at the Winter General Meeting of the American Institute of Electrical Engineers in New York City. In this paper a new and quite different approach to the problem was presented which the author chose to call the "Elimination Method". In this method the Ward and Hale technique is followed to derive formulas for the voltage corrections which should be applied to each bus, neglecting the higher order terms. Then, using the elimination method, the set of equations so derived are solved simultaneously for the various correction factors which, in the case presented, were the voltage magnitude and phase angle corrections. In addition to computing simultaneous correction factors, acceleration factors were applied to each correction. The result was a marked decrease in the number of iterations although the time per iteration was increased. In one example given, a reduction of from 40 to 3 iterations was experienced whereas the time per iteration was about doubled.

In the same paper the author described a technique for applying the elimination and cyclic iteration in combination to the same solution. The result in this case was an improvement over the Ward and Hale method but the method could not be accelerated by any known means.

It would appear that these last two papers have contributed much to our knowledge of iterative techniques. While both need, and undoubtedly will receive, much study in the future, it appears that a rapidly converging method will ultimately result and the digital computer will be a tool which power system engineers will find both economical and accurate for many future load flow problems.

III. A SOLUTION TO THE POWER SYSTEM PROBLEM

A. General Solution in Terms of Terminal Admittances

Electric power systems are nearly always balanced three-phase systems when operating under normal, steady-state conditions. For this special case the three-phase system can be represented on a single phase or "per phase" basis with no loss in generality. The per phase basis is to be assumed throughout the calculations that follow.

Consider the n-port, passive network, S, the elements of which are all assumed to be within the box shown in Fig. 1. Only the terminals of particular interest are shown and these are designated 1, 2, 3, ---n. Positive convention for current direction is into the network as shown in the diagram. Voltages are considered as voltage rises from the reference node to the terminal in question as shown in the diagram by $\dot{E}_3$ with appropriate polarity. In order that phasors, such as current and voltage can be clearly distinguished from matrices, the phasor values each have a dot directly above the letter as in Fig. 1. All terminals should be thought of as two-terminal pairs, or ports, with the same reference terminal common to all.

Define the following.

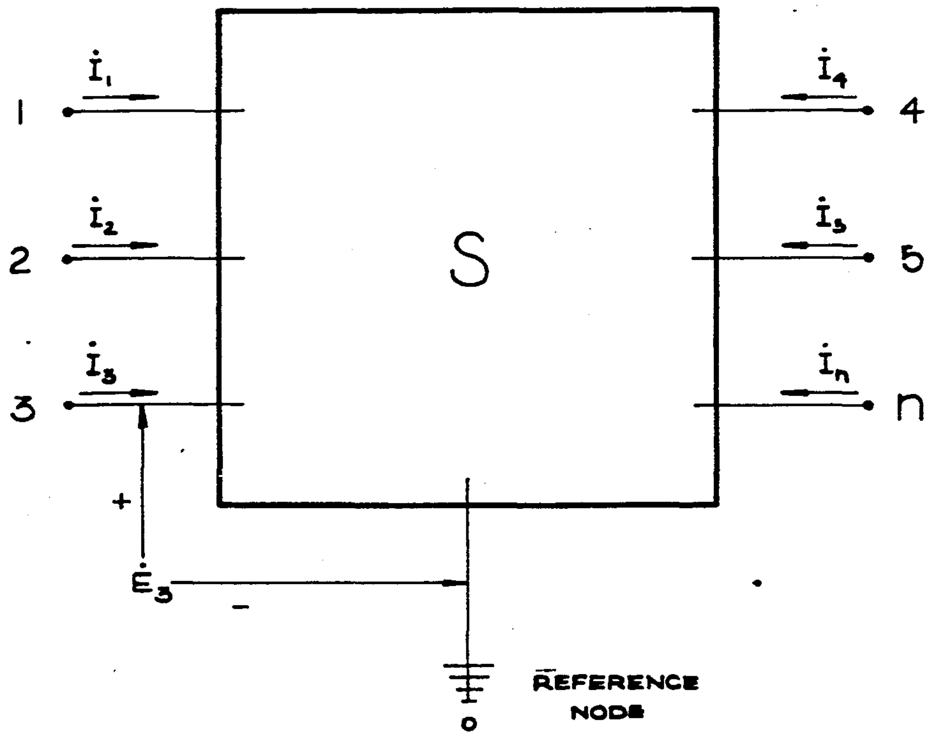
$$\dot{E}_k = e_k + jf_k \qquad 1$$

$$\dot{I}_k = a_k + jb_k \qquad 2$$

Using P and Q to represent power and reactive power respectively we may write

$$P_k + jQ_k = \dot{E}_k\dot{I}_k^* = (e_k a_k + f_k b_k) + j(f_k a_k - e_k b_k) \qquad 3$$

Fig. 1. The n-port system showing positive conventions for $\dot{I}_k$ and $\dot{E}_k$



$\dot{I}_1$
1
$\dot{I}_2$
2
$\dot{I}_3$
3
$\dot{I}_4$
4
$\dot{I}_5$
5
$\dot{I}_n$
n
S
+
$\dot{E}_3$
−
0
REFERENCE NODE

where (*) denotes the conjugate of the current is used. Then obviously

$$P_k = e_k a_k + f_k b_k \qquad 4$$

and $Q_k = f_k a_k - e_k b_k$.

Since the power and reactive power are dependent upon the network S, we require additional information to solve equations 4 completely. The additional constraints are supplied by Kirchhoff's law which may be stated as follows.

$$\begin{aligned}
\dot{I}_1 &= \dot{Y}_{11}\dot{E}_1 + \dot{Y}_{12}\dot{E}_2 + \cdots + \dot{Y}_{1n}\dot{E}_n \\
\dot{I}_2 &= \dot{Y}_{21}\dot{E}_1 + \dot{Y}_{22}\dot{E}_2 + \cdots + \dot{Y}_{2n}\dot{E}_n \\
\dot{I}_3 &= \dot{Y}_{31}\dot{E}_1 + \dot{Y}_{32}\dot{E}_2 + \cdots + \dot{Y}_{3n}\dot{E}_n \qquad 5 \\
&\text{------------------------------} \\
\dot{I}_n &= \dot{Y}_{n1}\dot{E}_1 + \dot{Y}_{n2}\dot{E}_2 + \cdots + \dot{Y}_{nn}\dot{E}_n
\end{aligned}$$

Here the symbol $\dot{Y}$ is a terminal admittance which may easily be related to the physical admittances within the network (19). It will now be helpful to expand $\dot{Y}$ into its real and quadrature components.

$$\dot{Y}_{kl} = G_{kl} + jB_{kl}$$

Now, expanding equations 5 in terms of the components we obtain the following.

$$\begin{aligned}
a_1 &= G_{11}e_1 - B_{11}f_1 + G_{12}e_2 - B_{12}f_2 + \cdots + G_{1n}e_n - B_{1n}f_n \\
a_2 &= G_{21}e_1 - B_{21}f_1 + G_{22}e_2 - B_{22}f_2 + \cdots + G_{2n}e_n - B_{2n}f_n \qquad 6 \\
&\text{--} \\
a_n &= G_{n1}e_1 - B_{n1}f_1 + G_{n2}e_2 - B_{n2}f_2 + \cdots + G_{nn}e_n - B_{nn}f_n
\end{aligned}$$

$$b_1 = B_{11}e_1 + G_{11}f_1 + B_{12}e_2 + G_{12}f_2 + \cdots + B_{1n}e_n + G_{1n}f_n$$

$$b_2 = B_{21}e_1 + G_{21}f_1 + B_{22}e_2 + G_{22}f_2 + \cdots + B_{2n}e_n + G_{2n}f_n \qquad 7$$

$$b_n = B_{n1}e_1 + G_{n1}f_1 + B_{n2}e_2 + G_{n2}f_2 + \cdots + B_{nn}e_n + G_{nn}f_n$$

This allows us to write the expressions for power and reactive power at the kth port in terms of only the voltages at that port.

$$P_k = e_k a_k + f_k b_k$$

$$= e_k(G_{k1}e_1 - B_{k1}f_1 + G_{k2}e_2 - B_{k2}f_2 + \cdots + G_{kn}e_n - B_{kn}f_n)$$

$$+ f_k(B_{k1}e_1 + G_{k1}f_1 + B_{k2}e_2 + G_{k2}f_2 + \cdots + B_{kn}e_n + G_{kn}f_n) \qquad 8$$

$$Q_k = f_k a_k - e_k b_k$$

$$= f_k(G_{k1}e_1 - B_{k1}f_1 + G_{k2}e_2 - B_{k2}f_2 + \cdots + G_{kn}e_n - B_{kn}f_n)$$

$$- e_k(B_{k1}e_1 + G_{k1}f_1 + B_{k2}e_2 + G_{k2}f_2 + \cdots + B_{kn}e_n + G_{kn}f_n) \qquad 9$$

We now have expanded the expressions for P and Q to the point where, if the voltages are all known, the power and reactive power can be found directly. This is a convenient way to express power system quantities since the voltage is usually known to within a few per cent and, even when iterative methods are required, the initial value chosen for the e_k and f_k can be relatively close. Rewriting equations 8 and 9 we obtain the following quadratic forms.

$$P_k = e_kG_{k1}e_1 - e_kB_{k1}f_1 + \cdots + e_kG_{kk}e_k - e_kB_{kk}f_k + \cdots$$

$$+ e_kG_{kn}e_n - e_kB_{kn}f_n + f_kB_{k1}e_1 + f_kG_{k1}f_1 + \cdots$$

$$+ f_kB_{kk}e_k + f_kG_{kk}f_k + \cdots + f_kB_{kn}e_n + f_kG_{kn}f_n \qquad 10$$

$$\begin{aligned} Q_k = {} & f_k G_{k1} e_1 - f_k B_{k1} f_1 + \cdots + f_k G_{kk} e_k - f_k B_{kk} f_k + \cdots \\ & + f_k G_{kn} e_n - f_k B_{kn} f_n - e_k B_{k1} e_1 - e_k G_{k1} f_1 - \cdots \\ & - e_k B_{kk} e_k - e_k G_{kk} f_k - \cdots - e_k B_{kn} e_n - e_k G_{kn} f_n \end{aligned} \tag{11}$$

1. Calculation of the A and B matrices

We now have the expressions for all electrical quantities usually desired at any port of an n-port system, namely current, power and reactive power. Furthermore, all of these quantities are expressed in terms of known system quantities and voltages. Thus, given the voltages, currents can be found from equations 5 or 6 and 7, power from 10 and reactive power from 11. The quantities of greatest interest to the power system engineer are the power and reactive power. It would therefore be to our advantage to be able to express power and reactive power in a simpler form than that given by equations 10 and 11. This can be accomplished quite easily in matrix notation and in a form so simple that it can easily be remembered.

Let the matrix A_k be defined as follows.

$$A_k = \left[\begin{array}{cccccccccc} 0 & 0 & --- & 0 & \frac{G_{k1}}{2} & \frac{B_{k1}}{2} & 0 & --- & 0 & 0 \\ 0 & 0 & --- & 0 & \frac{-B_{k1}}{2} & \frac{G_{k1}}{2} & 0 & --- & 0 & 0 \\ 0 & 0 & --- & 0 & \frac{G_{k2}}{2} & \frac{B_{k2}}{2} & 0 & --- & 0 & 0 \\ \hline 0 & 0 & --- & 0 & \frac{-B_{k,k-1}}{2} & \frac{G_{k,k-1}}{2} & 0 & --- & 0 & 0 \\ \frac{G_{k1}}{2} & \frac{-B_{k1}}{2} & --- & \frac{-B_{k,k-1}}{2} & G_{kk} & 0 & \frac{G_{k,k+1}}{2} & --- & \frac{G_{kn}}{2} & \frac{-B_{kn}}{2} \\ \frac{B_{k1}}{2} & \frac{G_{k1}}{2} & --- & \frac{G_{k,k-1}}{2} & 0 & G_{kk} & \frac{B_{k,k+1}}{2} & --- & \frac{B_{kn}}{2} & \frac{G_{kn}}{2} \\ 0 & 0 & --- & 0 & \frac{G_{k,k+1}}{2} & \frac{B_{k,k+1}}{2} & 0 & --- & 0 & 0 \\ \hline 0 & 0 & --- & 0 & \frac{-B_{kn}}{2} & \frac{G_{kn}}{2} & 0 & --- & 0 & 0 \end{array}\right] \tag{12}$$

Similarly, let B_k be defined as follows.

$$B_k = \left[\begin{array}{ccccccccc}
0 & 0 & --- & 0 & \frac{-B_{k1}}{2} & \frac{G_{k1}}{2} & 0 & --- & 0 \\
0 & 0 & --- & 0 & \frac{-G_{k1}}{2} & \frac{-B_{k1}}{2} & 0 & --- & 0 \\
0 & 0 & --- & 0 & \frac{-B_{k2}}{2} & \frac{G_{k2}}{2} & 0 & --- & 0 \\
\hline
0 & 0 & --- & 0 & \frac{-G_{k,k-1}}{2} & \frac{-B_{k,k-1}}{2} & 0 & --- & 0 \\
\frac{-B_{k1}}{2} & \frac{-G_{k1}}{2} & --- & \frac{-G_{k,k-1}}{2} & -B_{kk} & 0 & \frac{-B_{k,k+1}}{2} & --- & \frac{-G_{kn}}{2} \\
\frac{G_{k1}}{2} & \frac{-B_{k1}}{2} & --- & \frac{-B_{k,k-1}}{2} & 0 & -B_{kk} & \frac{G_{k,k+1}}{2} & --- & \frac{-B_{kn}}{2} \\
0 & 0 & --- & 0 & \frac{-B_{k,k+1}}{2} & \frac{G_{k,k+1}}{2} & 0 & --- & 0 \\
\hline
0 & 0 & --- & 0 & \frac{-G_{kn}}{2} & \frac{-B_{kn}}{2} & 0 & --- & 0
\end{array}\right] \qquad 13$$

Also define, using the prime symbol to indicate transpose, the following vector of voltage components.

$$\vec{x}' = (e_1, f_1, e_2, f_2, \cdot \cdot \cdot, e_n, f_n) \qquad 14$$

We can now express power and reactive power quite simply as follows.

$$P_k = \vec{x}' A_k \vec{x} \qquad 15$$

$$Q_k = \vec{x}' B_k \vec{x} \qquad 16$$

For clarification, the complete A and B matrices for a three-port network are shown in Appendix A. It will be noted that both matrices A and B are symmetric and, with a little practice, could easily be written from memory. Also, using the matrix notation for the quadratic form, the expansion to obtain power or reactive power on a digital computer are quite clear without any reference to details regarding the individual elements. It is only necessary to store the $\vec{x}$ vector and the A or B matrix in an orderly manner in the computer memory and expand directly with a minimum of computer orders. This form is also unique in that both power and reactive power may be determined directly without any reference whatsoever to the currents at each port.

2. Properties of the A and B matrices

It was mentioned previously that the A and B matrices are symmetric. It should also be noted that many of the elements are zero and that the zeros are grouped in blocks which are symmetric about the major diagonal. Thus, because of the zeros, any matrix multiplication involving these matrices will be greatly simplified if advantage is taken of their presence. Also, since the matrices are symmetric the eigenvalues of the matrices are all real and the linear spaces of eigenvectors may be spanned by real vectors (20).

Since the eigenvalues of a matrix reveal so clearly the properties of the matrix it would be helpful to know the eigenvalues of matrices A and B. Taking matrix A first, and defining λ to be an eigenvalue of A, we can write

$$A\vec{u} = \lambda\vec{u} \qquad 17$$

where $\vec{u}$ is a nonzero vector. Equation 17 can also be written in the form

$$(A - \lambda I)\vec{u} = 0 \qquad 18$$

where I is the identity matrix. Since matrix A is a 2n x 2n matrix, n being the number of ports, then I is also 2n x 2n and $\vec{u}$ is a 1 x 2n vector. Equation 18 has non-trivial solutions only if

$$\det(A - \lambda I) = 0. \tag{19}$$

At this point it will be easier to solve equation 19 for a specific matrix A of a given size. Appendix B gives the detailed calculations for cases with n equal to two and three. It appears obvious from an inspection of these results that A and B matrices of any size network will have two two-fold eigenvalues and the remaining $(2n - 4)$ eigenvalues will all be zero. Also, because of the nature of these eigenvalues, the eigenvalues of the matrices A_k and B_k for the kth port of an n-port network may be written by induction as follows. For A_k

$$\begin{aligned}\lambda_{Ak1} = \lambda_{Ak2} = \frac{G_{kk}}{2} + \Bigg[\left(\frac{G_{k1}}{2}\right)^2 &+ \left(\frac{G_{k2}}{2}\right)^2 + \cdots + \left(\frac{G_{kk}}{2}\right)^2 + \cdots \\ &+ \left(\frac{G_{kn}}{2}\right)^2 + \left(\frac{B_{k1}}{2}\right)^2 + \cdots + \left(\frac{B_{k,k-1}}{2}\right)^2 \\ &+ \left(\frac{B_{k,k+1}}{2}\right)^2 + \cdots + \left(\frac{B_{kn}}{2}\right)^2\Bigg]^{1/2}\end{aligned} \tag{20}$$

$$\begin{aligned}\lambda_{Ak3} = \lambda_{Ak4} = \frac{G_{kk}}{2} - \Bigg[\left(\frac{G_{k1}}{2}\right)^2 &+ \cdots + \left(\frac{G_{kk}}{2}\right)^2 + \cdots + \left(\frac{G_{kn}}{2}\right)^2 \\ &+ \left(\frac{B_{k1}}{2}\right)^2 + \cdots + \left(\frac{B_{k,k-1}}{2}\right)^2 + \left(\frac{B_{k,k+1}}{2}\right)^2 \\ &+ \cdots + \left(\frac{B_{kn}}{2}\right)^2\Bigg]^{1/2}\end{aligned} \tag{21}$$

$$\lambda_{Ak5} = \lambda_{Ak6} = \cdots = \lambda_{Ak,2n} = 0 \tag{22}$$

and for B_k

$$\lambda_{Bk1} = \lambda_{Bk2} = \frac{-B_{kk}}{2} + \left[\left(\frac{B_{k1}}{2}\right)^2 + \left(\frac{B_{k2}}{2}\right)^2 + \cdots + \left(\frac{B_{kk}}{2}\right)^2 + \cdots + \left(\frac{B_{kn}}{2}\right)^2 + \left(\frac{G_{k1}}{2}\right)^2 + \cdots + \left(\frac{G_{k,k-1}}{2}\right)^2 + \left(\frac{G_{k,k+1}}{2}\right)^2 + \cdots + \left(\frac{G_{kn}}{2}\right)^2\right]^{1/2} \qquad 23$$

$$\lambda_{Bk3} = \lambda_{Bk4} = \frac{-B_{kk}}{2} - \left[\left(\frac{B_{k1}}{2}\right)^2 + \left(\frac{B_{k2}}{2}\right)^2 + \cdots + \left(\frac{B_{kk}}{2}\right)^2 + \cdots + \left(\frac{B_{kn}}{2}\right)^2 + \left(\frac{G_{k1}}{2}\right)^2 + \cdots + \left(\frac{G_{k,k-1}}{2}\right)^2 + \left(\frac{G_{k,k+1}}{2}\right)^2 + \cdots + \left(\frac{G_{kn}}{2}\right)^2\right]^{1/2} \qquad 24$$

$$\lambda_{Bk5} = \lambda_{Bk6} = \cdots = \lambda_{Bk,2n} = 0. \qquad 25$$

Since the A and B matrices are symmetric the eigenvalues should be real. This is certainly true since all terms within the radical are squared terms.

It is also possible to draw some conclusions concerning the rank of the A and B matrices. Since any symmetric matrix may be transformed by an orthogonal matrix U such that U'AU is a diagonal matrix with each λ of A occupying one of the diagonal positions, then all A and B matrices have rank four. Since A and B are always of rank four the quadratic form $\vec{x}'A\vec{x}$ is also said to have rank four. We should also note that the eigenvalues of A and B always come in pairs. There are, therefore, two linearly independent eigenvectors associated with each eigenvalue.

One additional property of these matrices that requires investigation is that of definiteness. This is especially important in some iterative techniques where the form must be positive definite or positive semi-definite to guarantee convergence (18). An examination of the eigenvalues of A and B show that these matrices are certainly not definite. This agrees with physical reasoning since we know that P_k or Q_k may take on either positive, negative or zero values.

3. Properties of the elements of the A and B matrices

Now that the A and B matrices have been determined, some comments are in order regarding the individual elements of these matrices. These elements consist of the real or quadrature components of the terminal admittances. Kimbark (19) indicates the method of computing these admittances as, in fact, do many other texts in electrical engineering. In general, if we let $\dot{y}$ represent any physical admittance within the network, then

$$\dot{Y}_{kk} = \dot{y}_{k1} + \dot{y}_{k2} + \dot{y}_{k3} + \cdots + \dot{y}_{kn} + \dot{y}_{k0}$$

where 0 refers to the reference node and

$$\dot{Y}_{km} = -\dot{y}_{km} \quad \text{where } k \neq m.$$

These admittances are usually referred to as the self- and mutual-admittances respectively, or simply as the terminal admittances. Once the network impedances or admittances are known it is a simple matter to compute the terminal admittances.

One common problem which often arises in power system analysis is the handling of voltage transformations. Ward and Hale (6) have presented a convenient method for attacking this problem. The method will not be repeated here except to state that the turns ratio of the transformer effects both $\dot{Y}_{kk}$ at the node to which the transformer is connected and the $\dot{Y}_{km}$ of the transformer self admittance. These admittances can be calculated in terms of N, the turns ratio, and N can then be inserted as part of the data and altered as desired.

Another problem which is sometimes troublesome, especially in iterative solutions, is the presence of a node within the network which is not identified as a terminal. Two methods of handling such nodes are available. First, the node can be considered as a load terminal with zero power and reactive power as boundary conditions. The only trouble with this method is that the boundary conditions are satisfied in two different ways. One solution gives the correct voltage at all nodes such that zero power and reactive power results. The other solution is that of zero voltage, which is an unwanted solution. Where the presence of such an internal node is troublesome a second method of attack can be recommended without reservation. This is the elimination of the unwanted node by means of a star-mesh transformation.

B. Boundary Conditions in Power System Problems

Before proceeding with the solution of the network, the boundary conditions must be assigned. Since there is no unique set of boundary conditions for problems of this type it is necessary that all the most likely boundary conditions be examined so that some general conclusions may be drawn regarding the consistency of the equations to be solved.

1. Rules for determining a consistent set of boundary conditions

Consider the n-port network of Fig. 1 with current, voltage, power and reactive power defined as before in equations 1, 2 and 4 respectively. Recall also that we seek a solution for this network in terms of the rectangular form of the voltages or currents. We now make the following observations:

(1) For an n-port network there are 4n variables, namely a_k, b_k, e_k and f_k for $k = 1, 2, 3, \cdots n$;

(2) In order to express the problem mathematically so that one or more solutions are possible, exactly half of the 4n variables, or 2n must be specified and 2n equations must be written, or;

(3) If, for some reason we are unable to specify 2n variables explicitly, 2n independent constraints must be specified which are functions of the 4n variables, and 2n equations must be written.

Actually, in power system solutions it is not usually possible to follow observation 2 in defining the problem so constraints as suggested in observation 3 must be used.

Some commonly used boundary conditions for power systems are as follows, for quantities at the kth port.

Given P_k and Q_k

$$P_k = e_k a_k + f_k b_k$$
$$Q_k = f_k a_k - e_k b_k \qquad 26$$

Given P_k and $|\dot{E}_k|$

$$P_k = e_k a_k + f_k b_k$$
$$|\dot{E}_k| = (e_k^2 + f_k^2)^{1/2} \qquad 27$$

Given Q_k and $|\dot{E}_k|$

$$Q_k = f_k a_k - e_k b_k$$

$$|\dot{E}_k| = (e_k{}^2 + f_k{}^2)^{1/2} \qquad 28$$

Given $|\dot{E}_k|$ and δ_k

$$|\dot{E}_k| = (e_k{}^2 + f_k{}^2)^{1/2}$$

$$\delta_k = \tan^{-1} \frac{e_k}{f_k} \qquad 29$$

Other equally good boundary conditions which are not common in power system analysis are as follows.

Given $\dot{I}_k = a_k + jb_k$ 30

Given $\dot{E}_k = e_k + jf_k$ 31

Given $|\dot{I}_k|$ and φ_k

$$|\dot{I}_k| = (a_k{}^2 + b_k{}^2)^{1/2} \qquad 32$$

$$\varphi = \tan^{-1} \frac{b_k}{a_k}$$

Given $|\dot{I}_k|$ and δ_k

$$|\dot{I}_k| = (a_k{}^2 + b_k{}^2)^{1/2}$$

$$\delta_k = \tan^{-1} \frac{f_k}{e_k} \qquad 33$$

Still others could be devised which may be of interest in some particular application.

Let us examine the nature of the first four constraints, equations 26, 27, 28 and 29 since these are of particular interest in the power system problem. At load terminals it is common to use boundary conditions as in equations 26. These two equations in four unknowns have a doubly infinite set of solutions. However they place two constraints on our solution and, since 2n are needed in all, we need to find an additional (2n - 2) constraints from an examination of the boundary conditions at the remaining (n - 1) ports. At generator terminals, a common form of boundary condition is that given by equations 27. Thus each generator terminal, when specified as in equations 27, places two constraints upon the solution.

Now let us assume that (n - 1) ports of the n-port network are either load or generator ports. Both load and generator terminations, with boundary conditions as in equations 26 and 27, provide two constraints per port or $2(n - 1) = (2n - 2)$ in all. Since 2n are required, two more constraints must be available from a knowledge of the boundary conditions at the last port. It is at this point that one could get into real trouble in assigning boundary conditions, for this nth port is particularly critical. For now, let us be satisfied by stating that this port, usually called the swing generator, is usually constrained as in equations 29. Since equations 29 completely specify the voltage of this port, we may assume that δ_k is zero for this one terminal with no loss in generality. In many computations this is done for simplicity. It is also clear that the specifications of, say $|\dot{E}_k|$ alone, would not provide enough information since there would exist an infinite number of solutions corresponding to all possible values of δ_k. From a mathematical standpoint, it would have been equally correct to specify boundary conditions in equation 33 at this nth port. As a matter of fact, this would be the condition which would be chosen if we desired to hold the swing generator at a given value of current, for example at its maximum value. It should also be apparent that a specification as in equations 28 could have been used at any generator except the swing generator as these convey the same amount of information as equations 27, the usual generator boundary condition.

The foregoing suggests several other interesting possibilities which may sometimes be useful in solving a particular problem.

(1) It should be permissible to place three constraints on any one port if one constraint is removed from another port in the network.

(2) One port can be completely specified; i.e. a_k, b_k, e_k and f_k (four constraints) only if a total of two constraints are removed from the remaining (n - 1) ports.

(3) One half the ports may be completely specified (for n even) and the other half, although not assigned any boundary conditions, will be determined uniquely in the solution.

(4) The real part of $\dot{I}_k$ and $\dot{E}_k$, i.e. a_k and e_k respectively, may be specified at all ports and the quadrature components determined as a solution.

Actually conditions 1 and 2 are often useful in power system solutions. Conditions 3 and 4 are probably of academic interest only. All may easily be proven to be workable using an A-C Network Analyzer. The equations and solutions of some typical cases are given in Appendix C.

2. Boundary conditions at the nth port

It was suggested previously that the boundary conditions for the nth port may be troublesome. This assumes, of course that ports 1, 2, · · · (n - 1) are all load or generator ports with boundary conditions given as equations 26 or 27 above which are common for loads and generators, respectively. Suppose for example, port n were also constrained as a generator, using boundary condition 27. We now have a condition wherein the power, P_k, has been specified at all ports for $k = 1, 2, 3 \cdots n$. This being the case, we should be able to sum all the powers to obtain the net power entering the network. Using matrix notation, we may write;

$$P_L = \sum_{k=1}^{n} P_k = \sum_{k=1}^{n} \vec{x}' A_k \vec{x} = \vec{x}'(A_1 + A_2 + A_3 + \cdots + A_n)\vec{x}$$

or

$$P_L = \vec{x}' G_n \vec{x} \qquad 34$$

where

$$G_n = (A_1 + A_2 + A_3 + \cdots + A_n). \qquad 35$$

Then, performing the indicated addition,

$$G_n = \begin{bmatrix} G_{11} & 0 & G_{12} & 0 & ---- & G_{1n} & 0 \\ 0 & G_{11} & 0 & G_{12} & ---- & 0 & G_{1n} \\ G_{12} & 0 & G_{22} & 0 & ---- & G_{2n} & 0 \\ 0 & G_{12} & 0 & G_{22} & ---- & 0 & G_{2n} \\ \hline 0 & G_{1,n-1} & 0 & G_{2,n-1} & ---- & 0 & G_{n-1,n} \\ G_{1n} & 0 & G_{2n} & 0 & ---- & G_{nn} & 0 \\ 0 & G_{1n} & 0 & G_{2n} & ---- & 0 & G_{nn} \end{bmatrix}. \qquad 36$$

Now recall the way in which the diagonal elements of G_n were defined

$$G_{kk} = -(G_{k1} + G_{k2} + G_{k3} + \cdots + G_{kn} + G_{k0}) \qquad 37$$

where G_{k0} represents the conductance from k to 0, the reference node. It is apparent immediately that the diagonal terms, G_{11}, G_{22}, --- G_{nn} are at least as great as the negative of the sum of the remaining terms in each row. Stated another way,

$$G_{kk} = -\sum_{j=1}^{n}{}' G_{kj}$$

where the symbol $\sum'$ denotes the sum for all j except j = k. Bodewig (18, p. 79) proves that a matrix satisfying this condition is positive semi-definite. If G_n is positive semi-definite then

$$P_L = \vec{x}' G_n \vec{x} \geq 0 \qquad 38$$

or P_L is also positive semi-definite. Actually, our original premise that the network be passive requires this to be the case since no power can be generated in passive elements. It is interesting to note that there are two cases in which the equal sign of equation 38 applies: first, the trivial case with $\vec{x} = 0$ and second the case in which $\dot{E}_1 = \dot{E}_2 = \dot{E}_3 = \cdots = \dot{E}_n$.

Returning to our specification of the boundary condition at the nth port, it is now clear that the sum of all generated power must be greater than, or equal to, the sum of all load power. Should the power specified for the nth port be too small the solution may be nonexistent and if specified too large may give unrealistic voltages for power system operation. It is always safe to specify the voltage magnitude and phase angle for this port and this is the method usually used. Unless otherwise specified, this boundary condition will be assumed in what follows.

Before leaving this discussion of the nth port boundary condition, a question naturally arises regarding the reactive power losses. Following the same argument as before for real power losses, it is apparent that the S_k matrix analogous to G_k of equation 36 is not definite or semi-definite because B_{kj} may take either a positive or a negative sign for inductive or capacitive susceptance respectively ($k \neq j$) and the corresponding quadratic form for reactive power losses may be positive or negative. However should an examination of the network elements show, for example, that there are no capacitors involved, then the reactive losses are positive semi-definite. In power systems this will seldom be the case and, in view of the uncertainty as to the sign of reactive losses it would be better to avoid using reactive power as a boundary condition at every port in the network.

3. Recommended boundary conditions for power systems

Nearly all power system problems can be solved using boundary conditions as specified in equations 26, 27, 28 and 29 above. Also as previously mentioned, there is no loss in generality in assuming one of the voltages of the system to be coincident with the reference. In the solutions which follow therefore, the boundary conditions used will be as follows:

At load terminals

$$P_k = e_k a_k + f_k b_k \qquad\qquad 26$$
$$Q_k = f_k a_k - e_k b_k$$

At generator terminals

$$P_k = e_k a_k + f_k b_k \qquad\qquad 27$$
$$|\dot{E}_k| = (e_k^2 + f_k^2)^{1/2}$$

At synchronous condenser (or generator) terminals

$$Q_k = f_k a_k - e_k b_k$$
$$\left|\dot{E}_k\right| = (e_k^{\,2} + f_k^{\,2})^{1/2} \qquad 28$$

At the swing generator (the nth port)

$$\left|\dot{E}_k\right| = e_k$$
$$\delta_k = 0. \qquad 29a$$

C. Solution by Iterative Techniques

Now that the boundary conditions of the problem have been determined a solution for the n-port network may be attempted. Three methods of solution will be described in some detail. First, the Ward and Hale Iteration will be outlined because of its wide acceptance by the industry as a standard and because the technique used is typical of nearly all programs in use today for this problem. Next, two additional techniques will be presented and their effectiveness measured against the Ward and Hale method.

1. The Ward and Hale Iteration

This method was outlined briefly before and the salient features of the iteration are repeated here. The method makes use of the following relationships.

$$\dot{I}_k = \sum_{m=1}^{n} \dot{Y}_{km}\dot{E}_m = \sum_{m=1}^{n} (G_{km} + jB_{km})(e_m + jf_m) \qquad 39$$

$$\dot{I}_k = a_k + jb_k \qquad 40$$

$$P_k = e_k a_k + f_k b_k \qquad 41$$

$$Q_k = f_k a_k - e_k b_k \qquad 42$$

$$\left|\dot{E}_k\right| = (e_k{}^2 + f_k{}^2)^{1/2} \qquad 43$$

The iteration proceeds as follows:

(1) The voltages at all ports except that of the swing generator are estimated using some convenient value such as $1 + j0$. The voltage at the swing generator is completely specified and need not be changed as the iteration proceeds.

(2) The initial set of estimated voltages is used to compute $\dot{I}_1$ using equation 39. Current $\dot{I}_1$ is then used to compute P_1 using equation 41.

(3) The boundary conditions are given in terms of P_1 and Q_1 or P_1 and $\left|\dot{E}_1\right|$ depending on whether the port is a load or generator respectively. The calculated power is compared to the scheduled power and the reactive power or voltage magnitude is compared to the scheduled value to determine a correction for $\dot{E}_1$.

(4) Current $\dot{I}_2$ is now computed using equation 39 but using the corrected value for $\dot{E}_1$ just obtained.

(5) Step 3 is repeated to determine a correction for voltage $\dot{E}_2$, and so forth.

(6) The process is repeated until the voltage corrections become arbitrarily small.

The method of making the voltage corrections is also of interest. Consider first a load port with P_{ks} and Q_{ks} given as boundary conditions (the subscript s may be thought of as meaning "scheduled"). If we define error quantities as $\triangle P_k$ and $\triangle Q_k$, then

$$\begin{aligned} \triangle P_k &= P_{ks} - P_k \\ \triangle Q_k &= Q_{ks} - Q_k. \end{aligned} \qquad 44$$

Now assume a voltage correction is added to $\dot{E}_k$ and let this correction be

$$\triangle \dot{E}_k = \epsilon_k + j\,\xi_k$$

such that

$$P_{ks} + jQ_{ks} = (\dot{E}_k + \triangle \dot{E}_k)(\dot{I}_k + \dot{Y}_{kk} \triangle \dot{E}_k)^*. \qquad 45$$

Equation 45 can be solved to obtain

$$\begin{aligned} \triangle P_k &= \epsilon_k(e_k G_{kk} + f_k B_{kk} + a_k) + \xi_k(-e_k B_{kk} + f_k G_{kk} + b_k) \\ &\quad + G_{kk}(\epsilon_k^2 + \xi_k^2) \\ \triangle Q_k &= \epsilon_k(-e_k B_{kk} + f_k G_{kk} - b_k) + \xi_k(-e_k G_{kk} - f_k B_{kk} + a_k) \\ &\quad - B_{kk}(\epsilon_k^2 + \xi_k^2). \end{aligned} \qquad 46$$

Equations 46 are then solved for ϵ_k and ξ_k as a pair of linear equations by neglecting the higher order terms.

At generator terminals the same reasoning is followed to obtain the following pair of equations.

$$\begin{aligned} \triangle P_m &= \epsilon_m(e_m G_{mm} + f_m B_{mm} + a_m) + \xi_m(-e_m B_{mm} + f_m G_{mm} + b_m) \\ &\quad + G_{mm}(\epsilon_m^2 + \xi_m^2) \\ \triangle |\dot{E}_m|^2 &= |\dot{E}_m + \triangle \dot{E}_m|^2 - |\dot{E}_m|^2 \\ &= 2e_m \epsilon_m + 2f_m \xi_m + (\epsilon_m^2 + \xi_m^2). \end{aligned} \qquad 47$$

Equations 47 are solved as a linear set for ϵ_m and ξ_m by neglecting higher order terms.

Since it is the number of multiplications which largely determines the computer time, this is analyzed as follows where M signifies multiplications.

$$\begin{aligned} &\text{Load bus:} \qquad M = 4n + 17 \\ &\text{Generator bus:} \qquad M = 4n + 35 \end{aligned} \qquad 48$$

Here the square root routine required for generator ports is assumed equivalent to 20 multiplications. Since the number of loads is usually greater than the number of generators an average value of M would be about $4n + 20$ where n is the number of ports.

In a method such as this one it is not always clear what is meant by one iteration since this could be interpreted as one set of calculations at a port or one complete cycle of calculations at $n - 1$ ports. For comparison purposes with the methods which follow it is better to think of one iteration as being one set of calculations at one port. Then the total number of iterations will be the total of all such one-port calculations.

2. Iteration I

The Ward and Hale Iteration converges slowly by correcting the voltage of each port in a cyclic manner. It would appear feasible that convergence would be more rapid if corrections could be applied simultaneously to two or more ports. Iteration I is a method whereby corrections are applied simultaneously to all $n - 1$ unknown port voltages.

Let the subscript s indicate "scheduled" values as before. Furthermore define $\vec{X}$ to be the solution vector such that

$$P_{ks} = \vec{X}'A_k\vec{X}. \qquad 49$$

Then

$$P_k = P_{ks} - P_k = \vec{X}'A_k\vec{X} - \vec{x}'A_k\vec{x}; \qquad 50$$

and if we define $\vec{\epsilon}$ to be the error vector,

$$\vec{\lambda} = \vec{x} + \vec{\epsilon} \tag{51}$$

then

$$\triangle P_k = 2\vec{x}' A_k \vec{\epsilon} + \vec{\epsilon}' A_k \vec{\epsilon}. \tag{52}$$

Note that we may also write $\vec{\epsilon}$ as

$$\vec{\epsilon} = (\triangle_1, \triangle_2, \triangle_3, \cdots \triangle_{2n-2}, 0, 0) \tag{53}$$

where we are assuming port n to be the swing generator so that its voltage is completely specified and $\triangle_i$ is the correction required for each component of $\vec{x}$.

Similarly for reactive power constraints we may write

$$\triangle Q_k = 2\vec{x}' B_k \vec{\epsilon} + \vec{\epsilon}' B_k \vec{\epsilon}, \tag{54}$$

and for voltage magnitude constraints we get

$$\begin{aligned} \triangle \left|\dot{E}_k\right|^2 &= 2x_{2k-1}\triangle_{2k-1} + 2x_{2k}\triangle_{2k} + \triangle^2_{2k-1} + \triangle^2_{2k} \\ &= 2\vec{x}' M_k \vec{\epsilon} + \vec{\epsilon}' M_k \vec{\epsilon}. \end{aligned} \tag{55}$$

Here we have defined M_k to be

$$M_k = \begin{bmatrix} 0 - - - 0 & 0 - - - 0 \\ - - - - - - - & - - - - - - \\ 0 - - - 1 & 0 - - - 0 \\ 0 - - - 0 & 1 - - - 0 \\ - - - - - - - & - - - - - - \\ 0 - - - 0 & 0 - - - 0 \end{bmatrix} \tag{56}$$

and the ones appear only in locations $m_{2k-1,2k-1}$ and $m_{2k,2k}$.

If we neglect the second order corrections, or the terms similar to $\vec{\epsilon}' A_k \vec{\epsilon}$, we have a set of linear equations which may be solved simultaneously for $\vec{\epsilon}$. This set of equations is always 2n - 2 in number and 2n - 2 Δ_i corrections are found in each iteration.

A typical set of such correction equations given here for a three-port network (n = 3) where one port is a load, one a generator and the third is the swing generator.

$$\begin{aligned} \Delta P_1^{(m)} &= 2\vec{x}'^{(m)} A_1 \vec{\epsilon}^{(m+1)} \\ \Delta Q_1^{(m)} &= 2\vec{x}'^{(m)} B_1 \vec{\epsilon}^{(m+1)} \\ \Delta P_2^{(m)} &= 2\vec{x}'^{(m)} A_2 \vec{\epsilon}^{(m+1)} \\ \Delta \left|\dot{E}_2\right|^{2(m)} &= 2\vec{x}'^{(m)} M_2 \vec{\epsilon}^{(m+1)} \end{aligned} \qquad 57$$

The index (m) indicates the values used for the mth iteration.

The number of multiplications required for this method depends primarily upon the method used for solving the simultaneous linear equations for the correction vector. For example, a direct matrix inversion by the Sherman-Morrison and Bartlett method (18) requires $(2n - 2)^3 + (2n - 2)$ multiplications, where n here is the number of ports. Solution by the triangularization method (18) requires only $\frac{8}{3}n^3 - 6n^2$ multiplications for the complete solution. Since this savings is substantial it will be used for this problem and the total multiplications per iteration is approximately $\frac{8}{3}n^3 + 14n^2$ as compared with $4n + 20$ for the Ward and Hale method. Thus for n very large the savings in the number of iterations must be substantial if machine time is to be saved. Actually the solution of the linear equations, such as equation 57, for the correction vector might be improved by taking full advantage of the zeros in the coefficient matrix. This was not done in estimating the multiplications since the savings in multiplications should be partially offset by the increase in order coding which is subject to the skill of the coder, and the total number of zeros in a given matrix is a function of the physical network.

3. Iteration II

In Iteration I the following quadratic forms were developed from which the iteration scheme was evident.

$$\triangle P_k = 2\vec{x}'A_k\vec{\epsilon} + \vec{\epsilon}'A_k\vec{\epsilon} \qquad 52$$

$$\triangle Q_k = 2\vec{x}'B_k\vec{\epsilon} + \vec{\epsilon}'B_k\vec{\epsilon} \qquad 54$$

$$\triangle|\dot{E}_k|^2 = 2\vec{x}'M_k\vec{\epsilon} + \vec{\epsilon}'M_k\vec{\epsilon} \qquad 55$$

In Iteration I the method was based upon a linear solution of these quadratic forms by neglecting the higher order terms. Iteration II, on the other hand, is based upon using the higher order term as a correction to the above equations to obtain a better correction vector. Using the same three-port network as before to obtain equations analogous to equation 57, we have the following as our iteration scheme.

$$\begin{aligned}
\triangle P_1^{(m+1)} &= \triangle P_1^{(0)} - \vec{\epsilon}'^{(m)}A_1\vec{\epsilon}^{(m)} = 2\vec{x}'^{(0)}A_1\vec{\epsilon}^{(m+1)} \\
\triangle Q_1^{(m+1)} &= \triangle Q_1^{(0)} - \vec{\epsilon}'^{(m)}B_1\vec{\epsilon}^{(m)} = 2\vec{x}'^{(0)}B_1\vec{\epsilon}^{(m+1)} \\
\triangle P_2^{(m+1)} &= \triangle P_2^{(0)} - \vec{\epsilon}'^{(m)}A_2\vec{\epsilon}^{(m)} = 2\vec{x}'^{(0)}A_2\vec{\epsilon}^{(m+1)} \\
\triangle|\dot{E}_2|^{2(m+1)} &= \triangle|\dot{E}_2|^{2(0)} - \vec{\epsilon}'^{(m)}M_2\vec{\epsilon}^{(m)} = 2\vec{x}'^{(0)}M_2\vec{\epsilon}^{(m+1)}
\end{aligned} \qquad 58$$

The superscript (m) in this case refers to the iteration number. Here the error vector will be found as the simultaneous solution of the linear (in $\vec{\epsilon}^{(m+1)}$) system of equation 58.

Both Iterations I and II require the solution of a linear set of equations which we may describe as

$$C\vec{\triangle} = \vec{d} \qquad 59$$

where C is the coefficient matrix or the right-hand side of equations 57 or 58, $\vec{\triangle}$ is the vector of corrections and is the same as $\vec{\varepsilon}$ except the two zero corrections for the nth port voltage are missing. The vector $\vec{d}$ is the column of constants, or the left-hand side of equations 57 or 58. Note that in Iteration I the C matrix must be computed for each iteration. In Iteration II however, an initial computation is required to find $\triangle P_k^{(0)}$, $\triangle Q_k^{(0)}$ and $\triangle|\dot{E}_k|^{2(0)}$ using the initially estimated voltages $\vec{x}^{(0)}$. After this initial computation the C matrix is completely determined and is triangularized so that the iterations which follow alter only $\vec{d}$ and $\vec{\triangle}$. This results in a substantial savings in the number of multiplications, which are found to be $\frac{8}{3}n^3 + 14n^2 + (26n^2)N$ for this iteration, where N is the number of iterations. Notice that the n^3 term is required only once for making the initial computation and the multiplications of each iteration vary as n^2. This is a significant savings where n is large, but must still compete with the Ward and Hale Iteration where each iteration varies only as n to the first power.

4. Comparison of results

In order to obtain a direct comparison of the three methods and to prove the convergence of the last two, a sample problem was programmed all three ways. The problem chosen was that of Appendix A which shows a three-port network with one generator, one load and a swing generator. Actually a larger problem would be preferred but the three-port problem required nearly all of the 1024 words of storage of the Cyclone Computer, which was the machine used for the solution.

For each iteration method a family of runs was made using a different starting voltage vector for each run. The swing generator voltage was

Table 1. Comparison of results of three iteration methods

Run Number	Initial Voltage	Ward + Hale Iter.	Ward + Hale Mult.	Iteration I Iter.	Iteration I Mult.	Iteration II Iter.	Iteration II Mult.
1	1.0 + j0.0	15	480	5	990	9	2358
2	1.2 + j0.0	15	480	5	990	16	3942
3	1.4 + j0.0	15	480	5	990	22	5346
4	1.6 + j0.0	16	512	6	1188	28	6750
5	1.8 + j0.0	16	512	6	1188	34	8154
6	2.0 + j0.0	16	512	6	1188	40	9558
7	1.0 + j1.0	17	544	7	1386	F[b] - 65	15408
8	0.5 + j0.0	14	448	7	1386	D[c] - 7	1836
9	0.5 + j0.5	18	576	8	1584	D[c] - 7	1836
10[a]	1.0 + j0.0	F[b] - 110	3520	6	1188	D[c] - 11	2772

[a]On the 10th run, X_{23} was changed from +0.1 to -0.1.

[b]The notation F signifies "failed to converge".

[c]The notation D signifies "diverged".

completely specified to be 1.1 + j0 and the remaining two voltages were estimated initially at the same value, then this initial voltage varied over the range shown. The results are given in Table 1. In all runs the error voltage components were all compared with 10^{-8} and the iteration stopped when all components were less than this value. Details of the iterative solutions are given in Appendix D.

The results show clearly that, although the number of iterations in methods I and II may sometimes be less than those required by the Ward and Hale Iteration, the number of multiplications, or machine time, is much greater. This is a result of the n^3 and n^2 factors required for Iterations I and II multiplications whereas the Ward and Hale multiplications vary as n.

There are however, two factors which require comment in regard to the number of iterations and multiplications. One is the variation of the number of iterations with n and the other is the advantage gained by additional zeros in the A and B matrices. In regard to the former, Dr. Hale[1] has verified that the number of iterations for the Ward and Hale Iteration varies approximately as n^2. This is partly due to the greater number of iterations required to make one correction to each voltage since the ports are considered cyclically in the iteration process. In Iterations I and II this is not necessary as all are corrected simultaneously and the number of iterations should not be appreciably different from the example of Table 1 provided the initial estimates were equally close to the solution voltages. With regard to additional zeros in the A and B matrices, this becomes a function of the physical system under study. However it would be most uncommon for all transfer admittances to be present in any system. As a matter of fact the average number of transfer admittances associated with any one port would seldom exceed five or six regardless of the size of the system. A check of one of the largest systems in Iowa, for example, gave an average value of 2.5 transfer ad-

[1]Dr. H. W. Hale is a Professor of Electrical Engineering, Iowa State University, Ames, Iowa

mittances per port, a maximum number of five and n in this case was equal to 140. If full advantage is taken of these zeros in the matrix the number of multiplications will be greatly reduced. Systems larger than n = 3 were not attempted in this investigation because of the limited computer size. As was pointed out previously, the entire memory of the Cyclone was required to solve the three-port problem.

One definite disadvantage of Iterations I and II is the storage required for the A and B matrices which required $(2n)^2$ memory locations. Here again, a good coding technique would take advantage of the zeros and store only the necessary information.

5. Comments on elimination methods

The paper entitled "Elimination Methods for Load-Flow Studies" (17) presented in February 1961 was mentioned in the Review of Literature. The results of this paper parallel closely those of Iteration I presented here. The title suggests an "elimination" method was used in solving the linear equations for the error voltages and this was indeed the case. However, up to the point of simultaneous solution, the method is different. Instead of using matrix methods for this part of the solution, the authors used the conventional circuit techniques presented earlier by Ward and Hale. This paper also uses polar coordinates in the computations instead of rectangular coordinates which has the advantage of making it unnecessary to iterate to obtain voltage magnitudes where these are given as boundary conditions.

IV. CONCLUSIONS

The use of matrix calculus in the solution of the power system problems has several advantages. First, it provides a concise method of conveying the cumbersome mathematical details of the nonlinear equations which are to be solved. Second, once expressed in this new language, the equations can be more easily manipulated and several methods of iteration suggest themselves. It is quite probable, for example, that one would not immediately think of trying Iteration II had not the equations been so written, yet this iteration has some very desirable features not present in Iteration I.

The practical use to which the derived iterations may be put has yet to be proven. It is likely that their use will depend upon the system under consideration because of the desirability of having A and B matrices with many zeros. However, even in the worst possible case, as represented by the three-port example discussed previously, Iteration I converges at about the usual rate in spite of circuit changes for which the Ward and Hale method fails (see run 10, Table 1). Thus, it is probable that this iteration would find application in some systems which are not solvable by other techniques.

Another interesting application of matrix calculus is in the expression of losses in quadratic form. Here the evaluation of the G_n matrix shows it to be positive semi-definite, in agreement with physical knowledge of passive circuits, and places a definite restriction of the specification of boundary conditions for the problem. It is also evident from an examination of the S_n matrix exactly what circuit changes will make that matrix, and therefore the reactive power losses, positive semi-definite. Thus the use of a more concise mathematical language adds to our understanding of the problem in both a mathematical and a physical sense.

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VII. APPENDICES

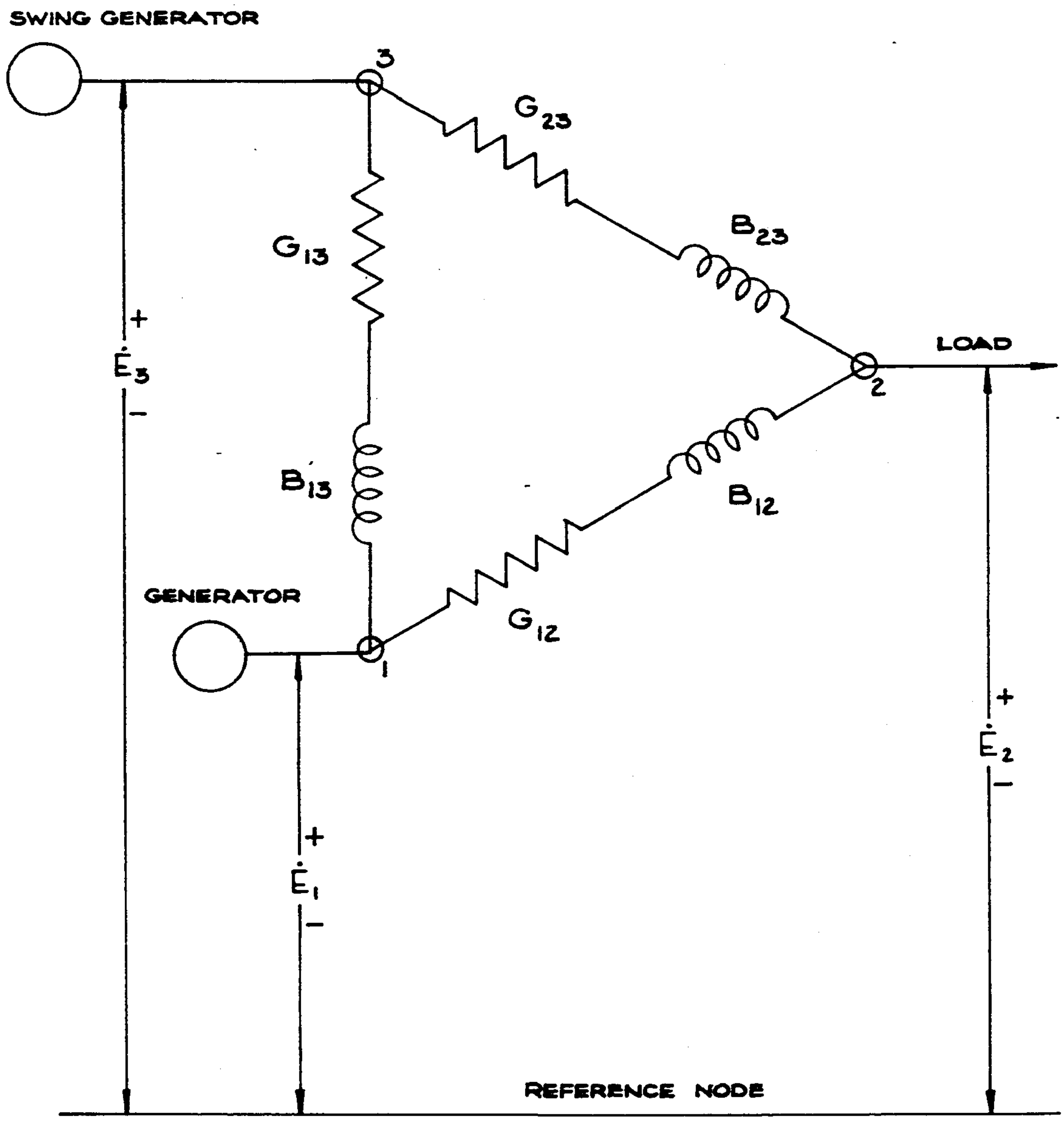
Appendix A. Matrices of a Three-Port Network

Consider the three-port network shown in Fig. 2 for which the A, B and M matrices are desired. These matrices may be written from an inspection of equations 12, 13 and 56 as follows.

$$A_1 = \begin{bmatrix} G_{11} & 0 & \frac{G_{12}}{2} & \frac{-B_{12}}{2} & \frac{G_{13}}{2} & \frac{-B_{13}}{2} \\ 0 & G_{11} & \frac{B_{12}}{2} & \frac{G_{12}}{2} & \frac{B_{13}}{2} & \frac{G_{13}}{2} \\ \frac{G_{12}}{2} & \frac{B_{12}}{2} & 0 & 0 & 0 & 0 \\ \frac{-B_{12}}{2} & \frac{G_{12}}{2} & 0 & 0 & 0 & 0 \\ \frac{G_{13}}{2} & \frac{B_{13}}{2} & 0 & 0 & 0 & 0 \\ \frac{-B_{13}}{2} & \frac{G_{13}}{2} & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (60)$$

$$A_2 = \begin{bmatrix} 0 & 0 & \frac{G_{12}}{2} & \frac{B_{12}}{2} & 0 & 0 \\ 0 & 0 & \frac{-B_{12}}{2} & \frac{G_{12}}{2} & 0 & 0 \\ \frac{G_{12}}{2} & \frac{-B_{12}}{2} & G_{22} & 0 & \frac{G_{23}}{2} & \frac{-B_{23}}{2} \\ \frac{B_{12}}{2} & \frac{G_{12}}{2} & & & \frac{B_{23}}{2} & \frac{G_{23}}{2} \\ 0 & 0 & \frac{G_{23}}{2} & \frac{B_{23}}{2} & 0 & 0 \\ 0 & 0 & \frac{-B_{23}}{2} & \frac{G_{23}}{2} & 0 & 0 \end{bmatrix} \qquad (61)$$

Fig. 2. Three-port network


SWING GENERATOR
3
G_{23}
G_{13}
B_{23}
+
$\dot{E}_3$
−
LOAD
2
B_{13}
B_{12}
GENERATOR
G_{12}
1
+
$\dot{E}_2$
−
+
$\dot{E}_1$
−
REFERENCE NODE

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{G_{13}}{2} & \frac{B_{13}}{2} \\ 0 & 0 & 0 & 0 & \frac{-B_{13}}{2} & \frac{G_{13}}{2} \\ 0 & 0 & 0 & 0 & \frac{G_{23}}{2} & \frac{B_{23}}{2} \\ 0 & 0 & 0 & 0 & \frac{-B_{23}}{2} & \frac{G_{23}}{2} \\ \frac{G_{13}}{2} & \frac{-B_{13}}{2} & \frac{G_{23}}{2} & \frac{-B_{23}}{2} & G_{33} & 0 \\ \frac{B_{13}}{2} & \frac{G_{13}}{2} & \frac{B_{23}}{2} & \frac{G_{23}}{2} & 0 & G_{33} \end{bmatrix} \qquad 62$$

$$B_1 = \begin{bmatrix} -B_{11} & 0 & \frac{-B_{12}}{2} & \frac{-G_{12}}{2} & \frac{-B_{13}}{2} & \frac{-G_{13}}{2} \\ 0 & -B_{11} & \frac{G_{12}}{2} & \frac{-B_{12}}{2} & \frac{G_{13}}{2} & \frac{-B_{13}}{2} \\ \frac{-B_{12}}{2} & \frac{G_{12}}{2} & 0 & 0 & 0 & 0 \\ \frac{-G_{12}}{2} & \frac{-B_{12}}{2} & 0 & 0 & 0 & 0 \\ \frac{-B_{13}}{2} & \frac{G_{13}}{2} & 0 & 0 & 0 & 0 \\ \frac{-G_{13}}{2} & \frac{-B_{13}}{2} & 0 & 0 & 0 & 0 \end{bmatrix} \qquad 63$$

$$B_2 = \begin{bmatrix} 0 & 0 & \frac{-B_{12}}{2} & \frac{G_{12}}{2} & 0 & 0 \\ 0 & 0 & \frac{-G_{12}}{2} & \frac{-B_{12}}{2} & 0 & 0 \\ \frac{-B_{12}}{2} & \frac{-G_{12}}{2} & -B_{22} & 0 & \frac{-B_{23}}{2} & \frac{-G_{23}}{2} \\ \frac{G_{12}}{2} & \frac{-B_{12}}{2} & 0 & -B_{22} & \frac{G_{23}}{2} & \frac{-B_{23}}{2} \\ 0 & 0 & \frac{-B_{23}}{2} & \frac{G_{23}}{2} & 0 & 0 \\ 0 & 0 & \frac{-G_{23}}{2} & \frac{-B_{23}}{2} & 0 & 0 \end{bmatrix} \qquad 64$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{-B_{13}}{2} & \frac{G_{13}}{2} \\ 0 & 0 & 0 & 0 & \frac{-G_{13}}{2} & \frac{-B_{13}}{2} \\ 0 & 0 & 0 & 0 & \frac{-B_{23}}{2} & \frac{G_{23}}{2} \\ 0 & 0 & 0 & 0 & \frac{-G_{23}}{2} & \frac{-B_{23}}{2} \\ \frac{-B_{13}}{2} & \frac{-G_{13}}{2} & \frac{-B_{23}}{2} & \frac{-G_{23}}{2} & -B_{33} & 0 \\ \frac{G_{13}}{2} & \frac{-B_{13}}{2} & \frac{G_{23}}{2} & \frac{-B_{23}}{2} & 0 & -B_{33} \end{bmatrix} \qquad 65$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad 66$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad 67$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad 68$$

We may then write

$$P_k = \vec{x}' A_k \vec{x},$$
$$Q_k = \vec{x}' B_k \vec{x} \qquad 69$$

and

$$\left|\dot{E}_k\right|^2 = \vec{x}' M_k \vec{x}.$$

For example,

$$P_2 = [e_1, f_1, e_2, f_2, e_3, f_3] \begin{bmatrix} 0 & 0 & \frac{G_{12}}{2} & \frac{B_{12}}{2} & 0 & 0 \\ 0 & 0 & \frac{-B_{12}}{2} & \frac{G_{12}}{2} & 0 & 0 \\ \frac{G_{12}}{2} & \frac{-B_{12}}{2} & G_{22} & 0 & \frac{G_{23}}{2} & \frac{-B_{23}}{2} \\ \frac{B_{12}}{2} & \frac{G_{12}}{2} & 0 & G_{22} & \frac{B_{23}}{2} & \frac{G_{23}}{2} \\ 0 & 0 & \frac{G_{23}}{2} & \frac{B_{23}}{2} & 0 & 0 \\ 0 & 0 & \frac{-B_{23}}{2} & \frac{G_{23}}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \\ e_2 \\ f_2 \\ e_3 \\ f_3 \end{bmatrix} \qquad 70$$

or

$$P_2 = G_{22}e_2{}^2 + G_{12}e_2e_1 + G_{23}e_2e_3 - B_{12}e_2f_1 - B_{23}e_2f_3$$
$$+ G_{22}f_2{}^2 + G_{12}f_2f_1 + G_{23}f_2f_3 + B_{12}f_2e_1 + B_{23}f_2e_3. \qquad 71$$

Appendix B. Eigenvalues and Eigenvectors of A and B Matrices

Consider first a two-port network with A and B matrices defined as in equations 12 and 13. Since we wish to find the eigenvalues of these matrices, we may proceed using equations 17, 18 and 19 as follows.

We have determined that

$$A_1 = \begin{bmatrix} G_{11} & 0 & \frac{G_{12}}{2} & \frac{-B_{12}}{2} \\ 0 & G_{11} & \frac{B_{12}}{2} & \frac{G_{12}}{2} \\ \frac{G_{12}}{2} & \frac{B_{12}}{2} & 0 & 0 \\ \frac{B_{12}}{2} & \frac{G_{12}}{2} & 0 & 0 \end{bmatrix}. \qquad 72$$

In order to simplify the algebra which will be required, let the following substitution be made.

$$G_{11} = G$$

$$\frac{G_{12}}{2} = m \qquad 73$$

$$\frac{B_{12}}{2} = n$$

Then

$$A_1 = \begin{bmatrix} G & 0 & m & -n \\ 0 & G & n & m \\ m & n & 0 & 0 \\ -n & m & 0 & 0 \end{bmatrix}. \qquad 74$$

Now set

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} G-\lambda & 0 & m & -n \\ 0 & G-\lambda & n & m \\ m & n & -\lambda & 0 \\ -n & m & 0 & -\lambda \end{bmatrix} = 0. \qquad 75$$

Expanding 75 we obtain the following polynomial in λ

$$\lambda^4 - 2G\lambda^3 + (G^2 - 2m^2 - 2n^2)\lambda^2 + 2G(m^2 + n^2)\lambda + (m^2 + n^2)^2 = 0 \qquad 76$$

which has the solution

$$\lambda = \frac{G}{2} \pm \sqrt{\frac{G^2 + 4(m^2 + n^2)}{2}}. \qquad 77$$

We may therefore define the roots of A_1 as follows.

$$\lambda_{A11} = \lambda_{A12} = \frac{G_{11}}{2} + \sqrt{\left(\frac{G_{11}}{2}\right)^2 + \left(\frac{G_{12}}{2}\right)^2 + \left(\frac{B_{12}}{2}\right)^2} \qquad 78$$

$$\lambda_{A13} = \lambda_{A14} = \frac{G_{11}}{2} - \sqrt{\left(\frac{G_{11}}{2}\right)^2 + \left(\frac{G_{12}}{2}\right)^2 + \left(\frac{B_{12}}{2}\right)^2}$$

In a similar manner we may show that the following are eigenvalues of A_2.

$$\lambda_{A21} = \lambda_{A22} = \frac{G_{22}}{2} + \sqrt{\left(\frac{G_{21}}{2}\right)^2 + \left(\frac{G_{22}}{2}\right)^2 + \left(\frac{B_{21}}{2}\right)^2}$$

$$\lambda_{A23} = \lambda_{A24} = \frac{G_{22}}{2} - \sqrt{\left(\frac{G_{21}}{2}\right)^2 + \left(\frac{G_{22}}{2}\right)^2 + \left(\frac{B_{21}}{2}\right)^2} \qquad 79$$

Then for matrix B_1,

$$\lambda_{B11} = \lambda_{B12} = \frac{-B_{11}}{2} + \sqrt{\left(\frac{B_{11}}{2}\right)^2 + \left(\frac{B_{12}}{2}\right)^2 + \left(\frac{G_{12}}{2}\right)^2}$$

$$\lambda_{B13} = \lambda_{B14} = \frac{-B_{11}}{2} - \sqrt{\left(\frac{B_{11}}{2}\right)^2 + \left(\frac{B_{12}}{2}\right)^2 + \left(\frac{G_{12}}{2}\right)^2} \qquad 80$$

and for matrix B_2,

$$\lambda_{B21} = \lambda_{B22} = \frac{-B_{22}}{2} + \sqrt{\left(\frac{B_{21}}{2}\right)^2 + \left(\frac{B_{22}}{2}\right)^2 + \left(\frac{G_{21}}{2}\right)^2}$$

$$\lambda_{B23} = \lambda_{B24} = \frac{-B_{22}}{2} - \sqrt{\left(\frac{B_{21}}{2}\right)^2 + \left(\frac{B_{22}}{2}\right)^2 + \left(\frac{G_{21}}{2}\right)^2} \qquad 81$$

completing the set of eigenvalues for the two-port network.

In exactly the same manner the eigenvalues for the three-port network of Fig. 2 may be determined. Since the details are repetitious and the form of these values is quite clear, only the eigenvalues of A_1 are given here.

$$\lambda_{A11} = \lambda_{A12} = \frac{G_{11}}{2} + \sqrt{\left(\frac{G_{11}}{2}\right)^2 + \left(\frac{G_{12}}{2}\right)^2 + \left(\frac{G_{13}}{2}\right)^2 + \left(\frac{B_{12}}{2}\right)^2 + \left(\frac{B_{13}}{2}\right)^2}$$

$$\lambda_{A13} = \lambda_{A14} = \frac{G_{11}}{2} - \sqrt{\left(\frac{G_{11}}{2}\right)^2 + \left(\frac{G_{12}}{2}\right)^2 + \left(\frac{G_{13}}{2}\right)^2 + \left(\frac{B_{12}}{2}\right)^2 + \left(\frac{B_{13}}{2}\right)^2} \qquad 82$$

$$\lambda_{A15} = \lambda_{A16} = 0$$

To determine the eigenvectors of these matrices we proceed as follows. Assume, for the two-port A_1 matrix, that $\lambda = \lambda_{A11} = \lambda_{A12}$. Then write

$$(A_1 - \lambda_{A11} I)\vec{y} = \vec{0}.$$

From this equation we may set up the following equations:

$$\begin{aligned} \lambda_{A13}y_1 + my_3 - ny_4 &= 0 \\ \lambda_{A13}y_2 + ny_3 + my_4 &= 0 \\ my_1 + ny_2 - \lambda_{A11}y_3 &= 0 \\ -ny_1 + my_2 - \lambda_{A11}y_4 &= 0 \end{aligned} \qquad 83$$

where the quantities m and n are as defined in equation 73. The coefficient matrix of 83 has rank two, therefore this set of equations has a nontrivial solution. Also, because the eigenvalues are two-fold values there are two linearly independent eigenvectors associated with each eigenvalue. One possible set of eigenvectors satisfying 83 are given in the matrix U_{A1} below as

$$U_{A1} = \frac{1}{v_{A1}} \begin{bmatrix} m & n & -\lambda_{A13} & 0 \\ n & -m & 0 & \lambda_{A13} \\ -\lambda_{A13} & 0 & -m & n \\ 0 & \lambda_{A13} & n & m \end{bmatrix} \qquad 84$$

where

$$v_{A1} = \lambda_{A13}(\lambda_{A13} - \lambda_{A11}). \tag{85}$$

It may be easily verified that U_{A1} is orthogonal and that

$$U'_{A1}A_1U_{A1} = D(\lambda_{A11}, \lambda_{A12}, \lambda_{A13}, \lambda_{A14}) \tag{86}$$

where D signifies a diagonal matrix.

For the two-port A_2, B_1 and B_2 matrices respectively, we may compute the following.

$$U_{A2} = \frac{1}{v_{A2}}\begin{bmatrix} -m & n & -\lambda_{A23} & 0 \\ n & m & 0 & \lambda_{A23} \\ -\lambda_{A23} & 0 & m & n \\ 0 & \lambda_{A23} & n & -m \end{bmatrix} \tag{87}$$

$$v_{A2} = \lambda_{A23}(\lambda_{A23} - \lambda_{A21}) \tag{88}$$

$$U_{B1} = \frac{1}{v_{B1}}\begin{bmatrix} -n & m & -\lambda_{B13} & 0 \\ m & n & 0 & \lambda_{B13} \\ -\lambda_{B13} & 0 & n & m \\ 0 & \lambda_{B13} & m & -n \end{bmatrix} \tag{89}$$

$$v_{B1} = \lambda_{B13}(\lambda_{B13} - \lambda_{B11}) \tag{90}$$

$$U_{B2} = \frac{1}{v_{B2}}\begin{bmatrix} n & m & -\lambda_{B23} & 0 \\ m & -n & 0 & \lambda_{B23} \\ -\lambda_{B23} & 0 & -n & m \\ 0 & \lambda_{B23} & m & n \end{bmatrix} \tag{91}$$

$$v_{B2} = \lambda_{B23}(\lambda_{B23} - \lambda_{B21}) \tag{92}$$

Note that in all of these matrices, m and n are defined as in equations 73.

It is interesting to note that the matrices given in equations 84 through 92 can be used to transform a quadratic form into a canonical form as follows.

Let

$$\vec{x} = U_{A1}\vec{y}. \qquad 93$$

Then, for the two-port case,

$$P_1 = \vec{x}'A_1\vec{x} = \vec{y}'U'_{A1}A_1U_{A1}\vec{y}$$
$$= \lambda_{A11}y_1^{\ 2} + \lambda_{A12}y_2^{\ 2} + \lambda_{A13}y_3^{\ 2} + \lambda_{A14}y_4^{\ 2}. \qquad 94$$

It does not appear possible however, to use the same transformation 93 to simultaneously reduce two quadratic forms, say P_1 and P_2 to canonical forms such as 94.

Appendix C. Solutions of Networks with Unusual Boundary Conditions

The following are network analyzer solutions for the three-port network of Fig. 2 with unusual boundary conditions, similar to those discussed in paragraph III, B. In all the problems shown, the network is the same network with terminal admittances as given in Table 2.

Table 2. Terminal admittances for the three-port network of Fig. 2

Ports	Per Unit Mhos G_{jk}	B_{jk}
1-2	-1.923	9.615
2-3	-1.538	12.308
3-1	-0.676	4.054

1. Solution of the three-port network with normal boundary conditions

The "normal" boundary conditions are those as specified in equations 26, 27, 28 and 29. These are tabulated as follows, along with the network analyzer solution.

Table 3. Solution of network with normal boundary conditions

Port	Quantity	Boundary Conditions	Solution
1	P_1	2.0	2.00
	Q_1	--	0.26
	$\dot{E}_1$	1.05	$1.05\underline{/4^\circ}$
2	P_2	-3.0	-3.00
	Q_2	-1.5	-1.50
	$\dot{E}_2$	--	$0.97\underline{/-4^\circ}$
3	P_3	--	1.08
	Q_3	--	1.85
	$\dot{E}_3$	$1.1\underline{/0^\circ}$	$1.10\underline{/0^\circ}$

2. <u>Solution of the three-port network with unusual boundary conditions</u>

Following are four additional solutions to the same problem as in paragraph 1 with boundary conditions not normally encountered in power system problems.

First consider the case of three boundary conditions on one port. As previously discussed, it is necessary to remove one boundary condition from another port in the system.

Now consider a case with four boundary conditions on one port. Note that the total number of boundary conditions remains at six.

The next solution represents a case wherein all boundary conditions are confined to only two ports.

Table 4. Three-port network with three constraints on one port

Port	Quantity	Boundary Condition	Solution
1	P_1	2.0	2.00
	Q_1	--	1.00
	$\dot{E}_1$	--	$1.13\angle 2^\circ$
2	P_2	-3.0	-3.00
	Q_2	-1.5	-1.50
	$\dot{E}_2$	1.0	$1.00\angle -5^\circ$
3	P_3	--	1.05
	Q_3	--	1.08
	$\dot{E}_3$	$1.1\angle 0^\circ$	$1.10\angle 0^\circ$

Finally, consider a case wherein the real components of all voltages and currents are given and the quadrature components are desired. Since the boundary conditions here are all linear, the solution will likewise be linear. In this case the solution can be written and a set of three linear equations solved.

We have previously defined the following.

$$\dot{I}_k = a_k + jb_k$$

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$$\dot{E}_k = e_k + jf_k$$

Now, writing the Kirchhoff's law constraints, this time in terms of terminal impedances rather than terminal admittances, we obtain the following.

Table 5. Three-port network with four constraints on one port

Port	Quantity	Boundary Condition	Solution
1	P_1	2.0	2.00
	Q_1	--	1.00
	$\dot{E}_1$	--	$1.13\angle 7^\circ$
2	P_2	-3.0	-3.00
	Q_2	-1.5	-1.50
	$\dot{E}_2$	$1.0\angle 0$	$1.00\angle 0^\circ$
3	P_3	--	1.05
	Q_3	--	1.08
	$\dot{E}_3$	1.1	$1.10\angle 5^\circ$

$$\begin{aligned} \dot{E}_1 &= Z_{11}\dot{I}_1 + Z_{12}\dot{I}_2 + Z_{13}\dot{I}_3 \\ \dot{E}_2 &= Z_{21}\dot{I}_1 + Z_{22}\dot{I}_2 + Z_{23}\dot{I}_3 \\ \dot{E}_3 &= Z_{31}\dot{I}_1 + Z_{32}\dot{I}_2 + Z_{33}\dot{I}_3 \end{aligned} \tag{96}$$

Here the terminal impedances are defined in the usual way where

$$Z_{ik} = R_{ik} + jX_{ik}. \tag{97}$$

Now by expanding equations 96 into real and quadrature components as defined in 95 and 97, we obtain the following.

$$\begin{aligned} e_1 &= R_{11}a_1 - X_{11}(b_1) + R_{12}a_2 - X_{12}(b_2) + R_{13}a_3 - X_{13}(b_3) \\ e_2 &= R_{21}a_1 - X_{21}(b_1) + R_{22}a_2 - X_{22}(b_2) + R_{23}a_3 - X_{23}(b_3) \\ e_3 &= R_{31}a_1 - X_{31}(b_1) + R_{32}a_2 - X_{32}(b_2) + R_{33}a_3 - X_{33}(b_3) \end{aligned} \tag{98}$$

Table 6. Three-port network with boundary conditions confined to two ports

Port	Quantity	Boundary Condition	Solution
1	P_1	2.0	2.00
	Q_1	1.5	1.50
	$\dot{E}_1$	0°	$1.16\underline{/0^\circ}$
2	P_2	-3.0	-3.00
	Q_2	-1.5	-1.50
	$\dot{E}_2$	1.0	$1.00\underline{/-6^\circ}$
3	P_3	--	1.18
	Q_3	--	2.16
	$\dot{E}_3$	--	$1.08\underline{/-1^\circ}$

$$\begin{aligned}(f_1) &= R_{11}(b_1) + X_{11}a_1 + R_{12}(b_2) + X_{12}a_2 + R_{13}(b_3) + X_{13}a_3 \\ (f_2) &= R_{21}(b_1) + X_{21}a_1 + R_{22}(b_2) + X_{22}a_2 + R_{23}(b_3) + X_{23}a_3 \\ (f_3) &= R_{31}(b_1) + X_{31}a_1 + R_{32}(b_2) + X_{32}a_2 + R_{33}(b_3) + X_{33}a_3\end{aligned} \qquad 99$$

For clarity, the unknown quantities in equations 98 and 99 are shown in parentheses. It will now be observed that equations 98 may be solved directly for b_1, b_2 and b_3. These values may then be substituted into equations 99 to find f_1, f_2 and f_3. Thus, with linear boundary conditions, however unusual they may be, the network may be solved directly as linear equations.

Appendix D. Iterative Solutions

Table 7. Ward and Hale iterative solution

Iteration	e_1	f_1	e_2	f_2
0	1.0000	0.0000	1.0000	0.0000
1	1.0513	0.1377	0.9833	-0.0595
2	1.0469	0.0922	0.9758	-0.0783
3	1.0472	0.0779	0.9751	-0.0841
4	1.0474	0.0737	0.9751	-0.0857
5	1.0475	0.0724	0.9751	-0.0862
6	1.0475	0.0721	0.9751	-0.0863
7-15	1.0475	0.0720	0.9751	-0.0864

Table 8. Iteration I solution

Iteration	e_1	f_1	e_2	f_2
0	1.0000	0.0000	1.0000	0.0000
1	1.0513	0.0784	0.9888	-0.0845
2	1.0476	0.0720	0.9753	-0.0864
3-5	1.0475	0.0720	0.9751	-0.0864

In order to illustrate the speed of convergence of the three methods investigated, an example is tabulated in Tables 7, 8 and 9 for all three methods. The problem used for this purpose is Run 1 of Table 1. This problem was programmed to iterate until each correction term was less than 10^{-8} and the solution was printed out to nine significant figures. Since

Fig. 3. Flow chart for Ward and Hale Iteration

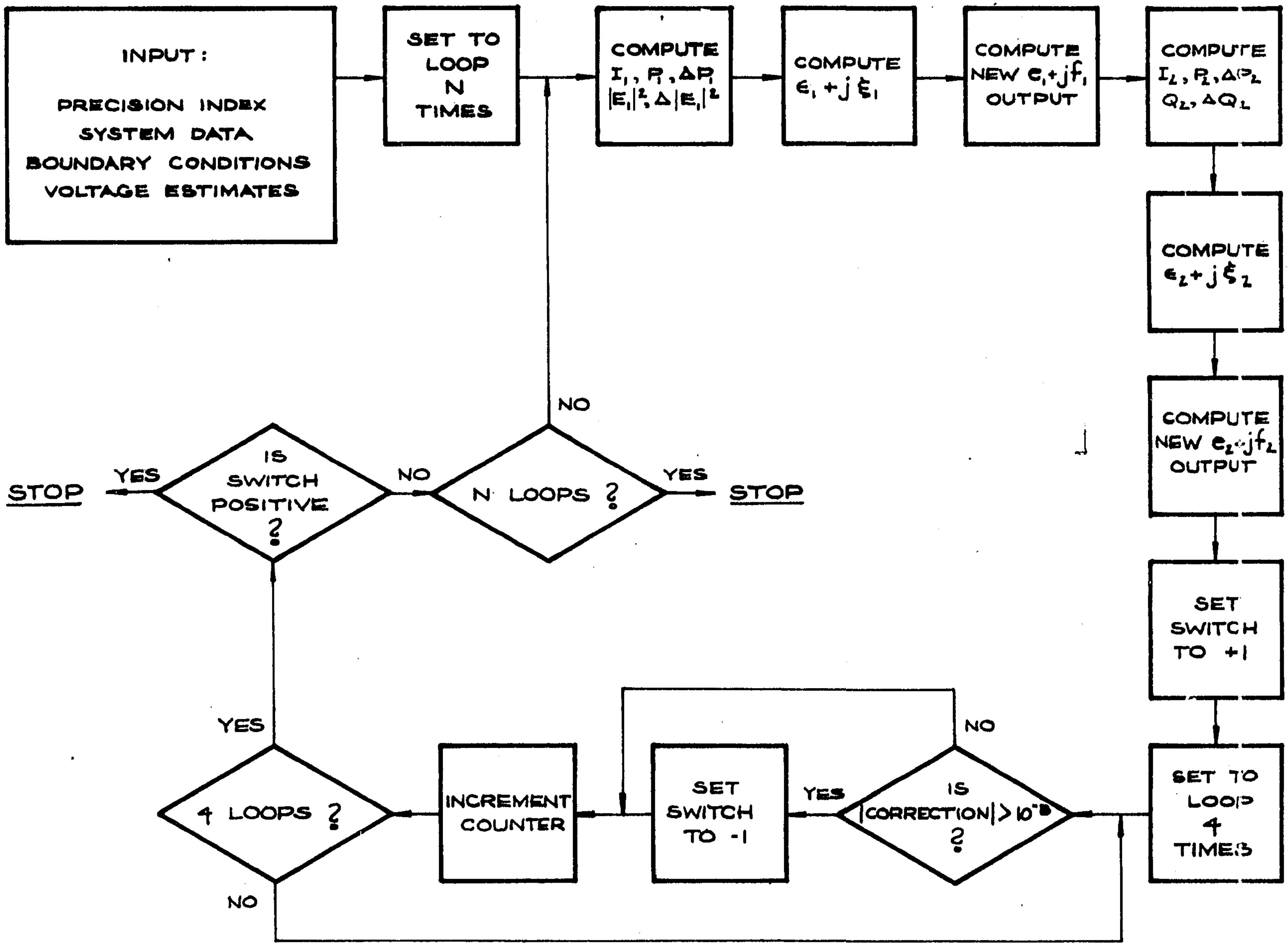
INPUT:
PRECISION INDEX
SYSTEM DATA
BOUNDARY CONDITIONS
VOLTAGE ESTIMATES
SET TO LOOP N TIMES
COMPUTE $I_1, P_1, \Delta P_1$ $|E_1|^2, \Delta|E_1|^2$
COMPUTE $\epsilon_1 + j\xi_1$
COMPUTE NEW $e_1 + jf_1$ OUTPUT
COMPUTE $I_2, P_2, \Delta P_2$ $Q_2, \Delta Q_2$
COMPUTE $\epsilon_2 + j\xi_2$
COMPUTE NEW $e_2 + jf_2$ OUTPUT
SET SWITCH TO +1
SET TO LOOP 4 TIMES
IS |CORRECTION| > 10^{-8} ?
NO
YES
SET SWITCH TO -1
INCREMENT COUNTER
4 LOOPS ?
YES
NO
IS SWITCH POSITIVE ?
YES
STOP
NO
N LOOPS ?
NO
YES
STOP

Fig. 4. Flow chart for Iteration I

Fig. 4. Flow chart for Iteration I

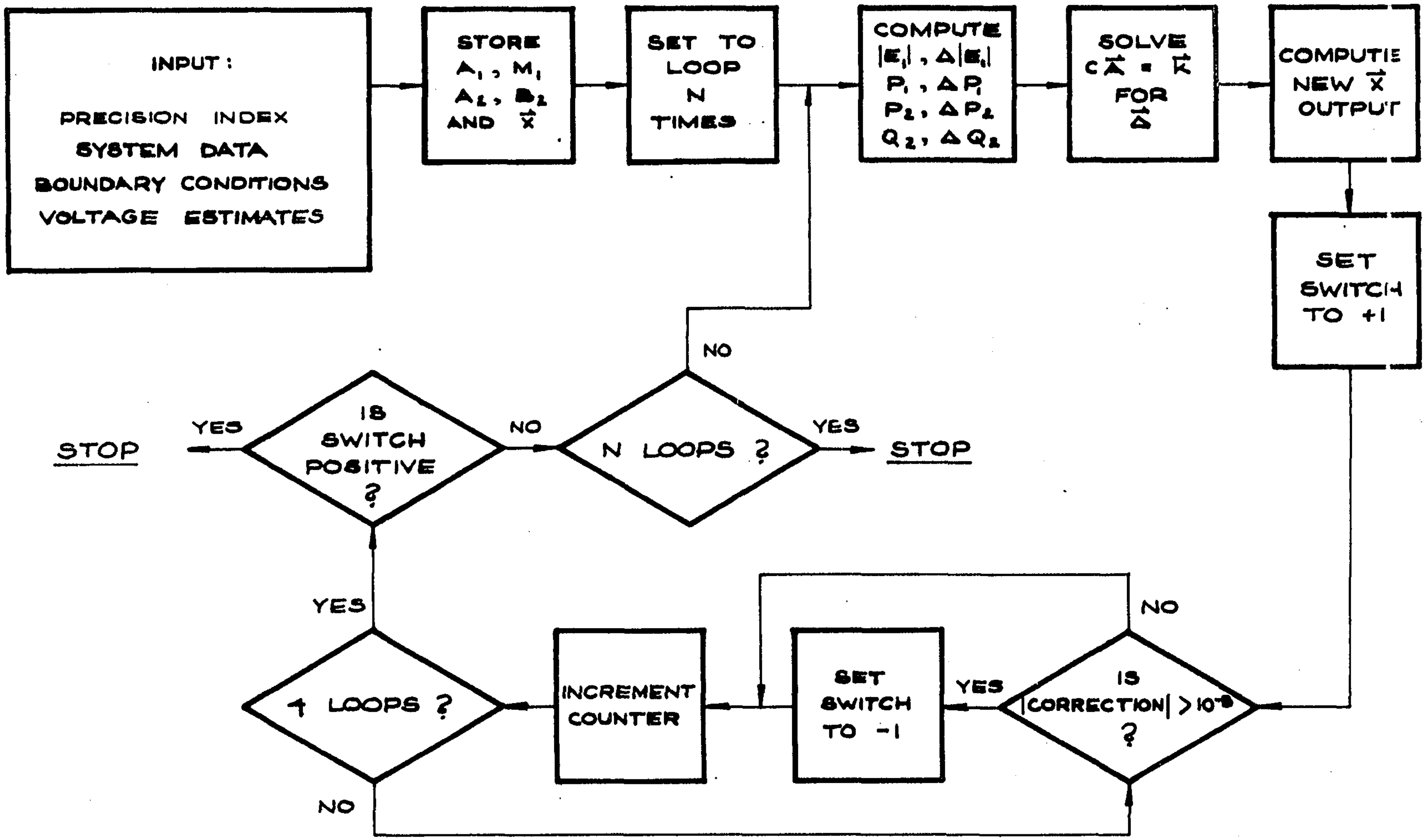
INPUT :
PRECISION INDEX
SYSTEM DATA
BOUNDARY CONDITIONS
VOLTAGE ESTIMATES
STORE
A1 , M1
A2 , B2
AND X
SET TO
LOOP
N
TIMES
COMPUTE
|E1| , Δ|E1|
P1 , ΔP1
P2 , ΔP2
Q2 , ΔQ2
SOLVE
CΔ = K
FOR
Δ
COMPUTE
NEW X
OUTPUT
SET
SWITCH
TO +1
NO
N LOOPS ?
YES
STOP
NO
IS
SWITCH
POSITIVE
?
YES
STOP
NO
IS
|CORRECTION| > 10
?
YES
SET
SWITCH
TO -1
INCREMENT
COUNTER
4 LOOPS ?
YES
NO

Fig. 5. Flow chart for Iteration II

Fig. 5. Flow chart for Iteration II

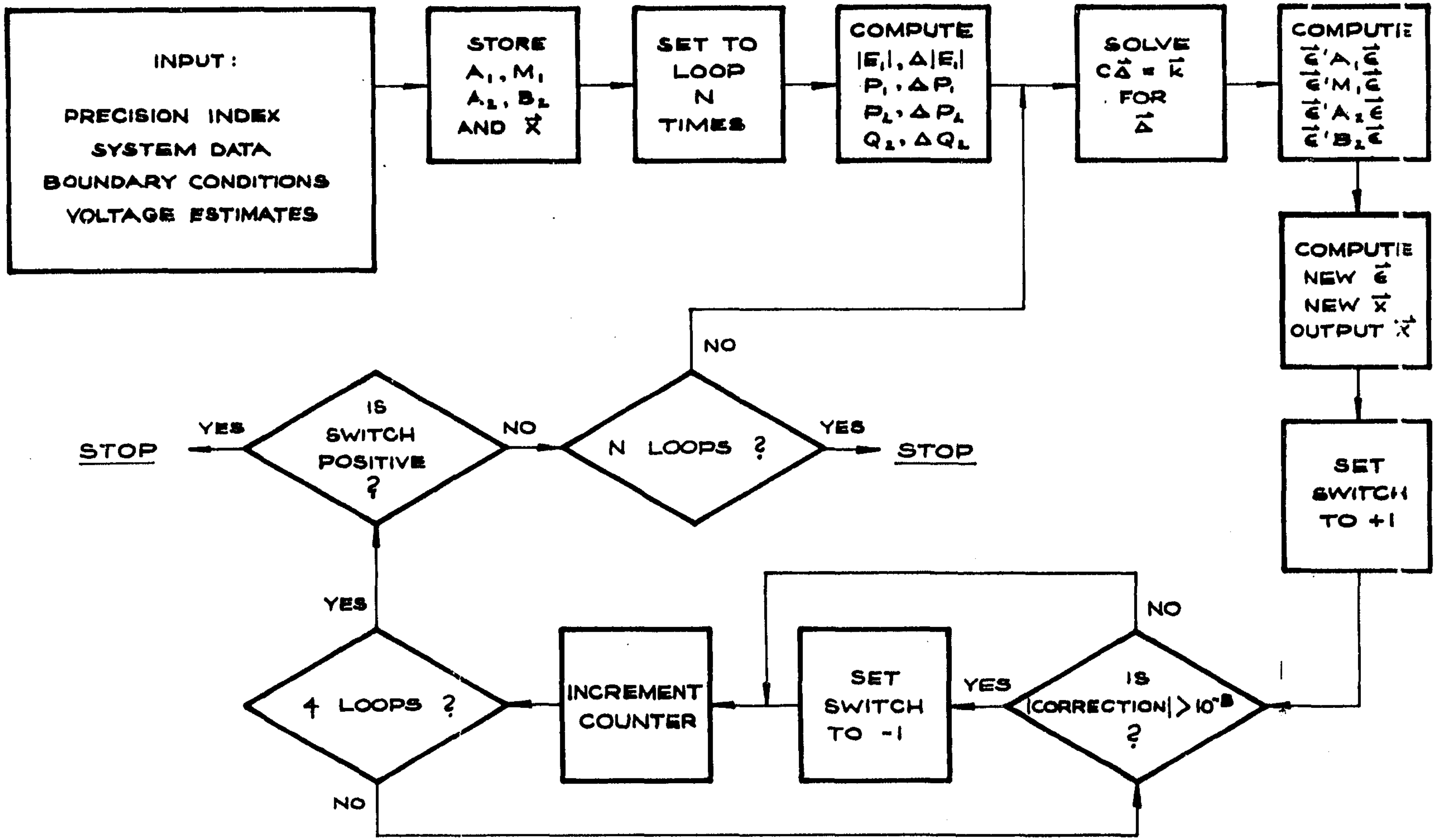
INPUT:
PRECISION INDEX
SYSTEM DATA
BOUNDARY CONDITIONS
VOLTAGE ESTIMATES
STORE A_1, M_1, A_2, B_2 AND $\vec{X}$
SET TO LOOP N TIMES
COMPUTE $|E_i|$, $\Delta|E_i|$, P_1, ΔP_1, P_2, ΔP_2, Q_2, ΔQ_2
SOLVE $C\vec{\Delta} = \vec{k}$ FOR $\vec{\Delta}$
COMPUTE $\vec{\epsilon}'A_1\vec{\epsilon}$, $\vec{\epsilon}'M_1\vec{\epsilon}$, $\vec{\epsilon}'A_2\vec{\epsilon}$, $\vec{\epsilon}'B_2\vec{\epsilon}$
COMPUTE NEW $\vec{\epsilon}$ NEW $\vec{x}$ OUTPUT $\vec{x}$
SET SWITCH TO +1
IS $|$CORRECTION$| > 10^{-8}$?
YES
NO
SET SWITCH TO -1
INCREMENT COUNTER
4 LOOPS ?
YES
NO
IS SWITCH POSITIVE ?
YES
STOP
NO
N LOOPS ?
NO
YES
STOP

Table 9. Iteration II solution

Iteration	e_1	f_1	e_2	f_2
0	1.0000	0.0000	1.0000	0.0000
1	1.0513	0.0784	0.9888	-0.0845
2	1.0469	0.0713	0.9757	-0.0862
3	1.0476	0.0720	0.9752	-0.0864
4-9	1.0475	0.0720	0.9751	-0.0864

the purpose of the tabulation here is to show only the nature of the convergence, the voltages have been rounded to four places. Note that for each iteration the initial voltage, called iteration zero, is 1.0 + j0.

The complete computer programs which solve the problem and output data in the manner tabulated above are not important since the program itself will probably be different if coded by different individuals. However, the flow charts from which the problems are coded are of interest since these charts allow us to visualize the complete solution regardless of the manner in which it is programmed. Flow charts for the three methods investigated are shown in Figs. 3, 4 and 5.